

Electric field in 3D gravity with torsion

M. Blagojević and B. Cvetković *

Institute of Physics, P. O. Box 57, 11001 Belgrade, Serbia

Abstract

It is shown that in static and spherically symmetric configurations of the system of Maxwell field coupled to 3D gravity with torsion, at least one of the Maxwell field components has to vanish. Restricting our attention to the electric sector of the theory, we find an interesting exact solution, corresponding to the azimuthal electric field. Its geometric structure is to a large extent influenced by the values of two different central charges, associated to the asymptotic AdS structure of spacetime.

1 Introduction

Three-dimensional (3D) general relativity (GR) has been used for nearly three decades as a theoretical laboratory for exploring basic features of the gravitational dynamics [1]. Among a number of outstanding results in this field, the discovery of the Bañados, Teitelboim and Zanelli (BTZ) black hole [2] was of particular importance, as it resulted in a significant influence on our understanding of the geometric and quantum structure of gravity.

In the early 1990s, Mielke and Baekler proposed a new geometric framework for 3D gravity, in which Riemannian geometry of spacetime was replaced by the more general, Riemann-Cartan geometry [3]. In this approach, the gravitational dynamics is characterized by both the torsion and the curvature [4, 5]. Recent developments along these lines reveal a respectable dynamical content of 3D gravity with torsion, characterized, in particular, by the existence of the conformal asymptotic structure with two different central charges, the BTZ black hole with torsion, the Chern-Simons formulation and the supersymmetric extension [6, 7, 8, 9].

The first electrically charged solution in Riemannian 3D gravity was found in [2]. Later studies of the problem led to a rather comprehensive analysis of the Einstein-Maxwell dynamics [10, 11, 12, 13, 14]. The purpose of the present paper is to start similar investigations in 3D gravity with torsion, with a focus on the electric sector of the theory.

The layout of the paper is as follows. In section 2, we derive the general field equations of the system of Maxwell field coupled to 3D gravity with torsion. In section 3, we use these equations to prove a specific no-go theorem, saying that in static and spherically symmetric field configurations, at least one component of the Maxwell field has to vanish. In section 4, we restrict our attention to the electric sector of the theory and show that,

*Email addresses: mb@phy.bg.ac.yu, cbranislav@phy.bg.ac.yu

in contrast to Riemannian theory, dynamically allowed configurations with a radial electric field are trivial, as the radial field has to be constant. In section 5, we find an interesting solution generated by the azimuthal electric field, a generalization of the solution found by Cataldo [14] in Riemannian GR with a cosmological constant. The solution is shown to have vanishing conserved charges (energy, angular momentum and electric charge), while its geometric structure is determined by the central charges, associated to the asymptotic AdS sector of spacetime. Finally, appendices contain some technical material.

Our conventions are given by the following rules: the Latin indices (i, j, k, \dots) refer to the local Lorentz frame, the Greek indices $(\mu, \nu, \lambda, \dots)$ refer to the coordinate frame, and both run over 0,1,2; the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric tensor ε^{ijk} and the related tensor density $\varepsilon^{\mu\nu\rho}$ are both normalized so that $\varepsilon^{012} = 1$.

2 Maxwell field in 3D gravity with torsion

Theory of gravity with torsion can be naturally described as a Poincaré gauge theory (PGT), with an underlying spacetime structure corresponding to Riemann-Cartan geometry [4, 5].

Basic gravitational variables in PGT are the triad field b^i and the Lorentz connection $A^{ij} = -A^{ji}$ (1-forms). The corresponding field strengths are the torsion and the curvature: $T^i := db^i + A^i_m \wedge b^m$, $R^{ij} := dA^{ij} + A^i_m \wedge A^{mj}$ (2-forms). In 3D, we can simplify the notation by introducing $A^{ij} =: -\varepsilon^{ij}_k \omega^k$ and $R^{ij} =: -\varepsilon^{ij}_k R^k$, which yields:

$$T^i = db^i + \varepsilon^i_{jk} \omega^j \wedge b^k, \quad R^i = d\omega^i + \frac{1}{2} \varepsilon^i_{jk} \omega^j \wedge \omega^k. \quad (2.1)$$

The covariant derivative $\nabla(\omega)$ acts on a general tangent-frame spinor/tensor in accordance with its spinorial/tensorial structure; when X is a form, $\nabla X := \nabla \wedge X$.

PGT is characterized by a useful identity:

$$\omega^i \equiv \tilde{\omega}^i + K^i, \quad (2.2a)$$

where $\tilde{\omega}^i$ is the Levi-Civita (Riemannian) connection, and K^i is the contortion 1-form, defined implicitly by

$$T^i =: \varepsilon^i_{mn} K^m \wedge b^n. \quad (2.2b)$$

Using this identity, one can express the curvature $R_i = R_i(\omega)$ in terms of its *Riemannian* piece $\tilde{R}_i = R_i(\tilde{\omega})$ and the contortion K_i :

$$2R_i \equiv 2\tilde{R}_i + 2\tilde{\nabla} K_i + \varepsilon_{imn} K^m \wedge K^n. \quad (2.2c)$$

The antisymmetry of the Lorentz connection A^{ij} implies that the geometric structure of PGT corresponds to Riemann-Cartan geometry, in which b^i is an orthonormal coframe, $g := \eta_{ij} b^i \otimes b^j$ is the metric of spacetime, and ω^i is the Cartan connection.

In local coordinates x^μ , we can write $b^i = b^i_\mu dx^\mu$, the frame dual to b^i reads $h_i = h_i^\mu \partial_\mu$ and satisfies the property $h_i \lrcorner b^j = h_i^\mu b^j_\mu = \delta^j_i$, where \lrcorner is the interior product. In what follows, we will omit the wedge product sign \wedge for simplicity.

Lagrangian and the field equations

General gravitational dynamics in Riemann-Cartan spacetime is determined by Lagrangians which are at most quadratic in field strengths. Omitting the quadratic terms, we arrive at the *topological* Mielke-Baekler (MB) model for 3D gravity [3]:

$$L_0 = 2ab^i R_i - \frac{\Lambda}{3} \varepsilon_{ijk} b^i b^j b^k + \alpha_3 L_{\text{CS}}(\omega) + \alpha_4 b^i T_i. \quad (2.3a)$$

Here, $a = 1/16\pi G$ and $L_{\text{CS}}(\omega)$ is the Chern-Simons Lagrangian for the Lorentz connection, $L_{\text{CS}}(\omega) = \omega^i d\omega_i + \frac{1}{3} \varepsilon_{ijk} \omega^i \omega^j \omega^k$. The MB model is a natural generalization of GR with a cosmological constant (GR_Λ).

The complete dynamics includes also the contribution of matter fields, minimally coupled to gravity. We focus our attention to the case when matter is represented by the Maxwell field:

$$L = L_0 + L_M, \quad L_M := -\frac{1}{4} F^* F, \quad (2.3b)$$

where $F = dA$.

By varying L with respect to b^i and ω^i , one obtains the gravitational field equations:

$$\begin{aligned} 2aR_i + 2\alpha_4 T_i - \Lambda \varepsilon_{ijk} b^j b^k &= \Theta_i, \\ 2\alpha_3 R_i + 2aT_i + \alpha_4 \varepsilon_{ijk} b^j b^k &= \Sigma_i, \end{aligned} \quad (2.4)$$

where $\Theta_i := -\delta L_M / \delta b^i$ and $\Sigma_i := -\delta L_M / \delta \omega^i$ are the energy-momentum and spin currents (2-forms) of matter. The Maxwell field currents are given by

$$\Theta_i = \frac{1}{2} [F(h_i \rfloor *F) - (h_i \rfloor F)^* F], \quad \Sigma_i = 0.$$

In the nondegenerate sector with $\Delta := \alpha_3 \alpha_4 - a^2 \neq 0$, these equations can be rewritten as

$$2T_i - p \varepsilon_{ijk} b^j b^k = u \Theta_i, \quad (2.5a)$$

$$2R_i - q \varepsilon_{ijk} b^j b^k = -v \Theta_i, \quad (2.5b)$$

where

$$\begin{aligned} p &:= \frac{\alpha_3 \Lambda + \alpha_4 a}{\Delta}, & q &:= -\frac{(\alpha_4)^2 + a \Lambda}{\Delta}, \\ u &:= \frac{\alpha_3}{\Delta}, & v &:= \frac{a}{\Delta}. \end{aligned}$$

Introducing the energy-momentum tensor of matter by $\mathcal{T}^k_i := *(b^k \Theta_i)$, the matter current Θ_i can be expressed as follows:

$$\Theta_i = \frac{1}{2} (\mathcal{T}^k_i \varepsilon_{kmn}) b^m b^n = \varepsilon_{imn} t^m b^n, \quad (2.6a)$$

$$t^m := - \left(\mathcal{T}^m_k - \frac{1}{2} \delta^m_k \mathcal{T} \right) b^k, \quad (2.6b)$$

where $\mathcal{T} = \mathcal{T}^k_k$. The Maxwell energy-momentum tensor reads:

$$\mathcal{T}^k_i = -F^{km} F_{im} + \frac{1}{4} \delta^k_i F^2, \quad (2.6c)$$

with $F^2 = F^{mn}F_{mn}$,

These results can be used to simplify the field equations (2.5). Indeed, if we substitute the above Θ_i into (2.5a) and compare the result with (2.2b), we find the following form of the contortion:

$$K^m = \frac{1}{2}(pb^m + ut^m). \quad (2.7a)$$

After that, we can rewrite the second field equation (2.5b) as

$$2R_i = q\varepsilon_{imn}b^mb^n - v\varepsilon_{imn}t^mb^n, \quad (2.7b)$$

where the Cartan curvature R_i is calculated using the identity (2.2c):

$$2R_i = 2\tilde{R}_i + u\tilde{\nabla}t_i + \varepsilon_{imn}\left(\frac{p^2}{4}b^mb^n + \frac{up}{2}t^mb^n + \frac{u^2}{4}t^mt^n\right). \quad (2.7c)$$

In this form of the gravitational field equations, the role of the Maxwell field as a source of gravity is clearly described by the 1-form t^i .

The Maxwell field equations take the standard form:

$$d^*F = 0. \quad (2.8)$$

These equations, together with a suitable set of boundary conditions, define the complete dynamics of both the gravitational and the Maxwell field.

3 A no-go theorem

In order to explore basic dynamical features of the system of Maxwell field coupled to 3D gravity with torsion, we begin by looking at *static* and *spherically symmetric* field configurations. Using the Schwarzschild-like coordinates $x^\mu = (t, r, \varphi)$, we make the following ansatz for the triad field,

$$b^0 = Ndt, \quad b^1 = B^{-1}dr, \quad b^2 = Kd\varphi, \quad (3.1)$$

and for the Maxwell field:

$$F = E_rb^0b^1 - Hb^1b^2 + E_\varphi b^2b^0. \quad (3.2)$$

Here, N, B, K and E_r, H, E_φ are the unknown functions of the radial coordinate r .

The Maxwell equations (2.8) read:

$$E'_r B + \gamma E_r = 0, \quad H'B + \alpha H = 0, \quad E_\varphi = E_\varphi(r), \quad (3.3a)$$

where α, γ are components of the Riemannian connection, defined in Appendix A, and E_φ remains an arbitrary function of r . The corresponding first integrals are

$$E_r K = Q_1, \quad H N = Q_3, \quad (3.3b)$$

where Q_1 and Q_3 are constants.

Next, we calculate the energy-momentum tensor

$$\mathcal{T}^i_j = \frac{1}{2} \begin{pmatrix} E_r^2 + E_\varphi^2 + H^2 & 2E_\varphi H & 2E_r H \\ -2E_\varphi H & E_r^2 - E_\varphi^2 - H^2 & -2E_r E_\varphi \\ -2E_r H & -2E_r E_\varphi & -E_r^2 + E_\varphi^2 - H^2 \end{pmatrix}, \quad (3.4)$$

and find the expression for t^i :

$$\begin{aligned} t^0 &= -\left(\frac{\mathcal{T}}{2} + H^2\right) b^0 - E_\varphi H b^1 - E_r H b^2, \\ t^1 &= E_\varphi H b^0 - \left(\frac{\mathcal{T}}{2} - E_\varphi^2\right) b^1 + E_r E_\varphi b^2, \\ t^2 &= E_r H b^0 + E_r E_\varphi b^1 - \left(\frac{\mathcal{T}}{2} - E_r^2\right) b^2, \end{aligned} \quad (3.5)$$

where $\mathcal{T} = \mathcal{T}^k_k = \frac{1}{2} (E_r^2 + E_\varphi^2 - H^2)$.

Technical details leading to the explicit form of the gravitational field equations (2.7b) are summarized in appendix A. All the components of these equations can be conveniently divided in two sets: those with $(i, m, n) = (0, 1, 2), (2, 0, 1), (1, 2, 0)$ are called diagonal, all the others are nondiagonal. Introducing

$$V := v + up/2$$

to simplify the notation, the nondiagonal equations read:

$$V E_r H - \frac{u^2}{4} E_r H \mathcal{T} = u \left(-\frac{1}{2} \gamma E_r^2 + \frac{1}{2} E_\varphi E'_\varphi B + \alpha E_\varphi^2 - \frac{1}{2} \alpha H^2 \right), \quad (3.6a)$$

$$V E_\varphi H - \frac{u^2}{4} E_\varphi H \mathcal{T} = -u E_r E_\varphi \alpha, \quad (3.6b)$$

$$V E_r E_\varphi - \frac{u^2}{4} E_r E_\varphi \mathcal{T} = u H (E'_\varphi B + \alpha E_\varphi), \quad (3.6c)$$

$$V E_\varphi H - \frac{u^2}{4} E_\varphi H \mathcal{T} = u E_r (E'_\varphi B + \gamma E_\varphi), \quad (3.6d)$$

$$V E_r H - \frac{u^2}{4} E_r H \mathcal{T} = -u \left(\frac{1}{2} \gamma E_r^2 + \frac{1}{2} E_\varphi E'_\varphi B + \gamma E_\varphi^2 + \frac{1}{2} \alpha H^2 \right), \quad (3.6e)$$

$$V E_r E_\varphi - \frac{u^2}{4} E_r E_\varphi \mathcal{T} = -u E_\varphi H \gamma, \quad (3.6f)$$

while the diagonal ones are:

$$-2(\gamma' B + \gamma^2) = 2\Lambda_{\text{eff}} - V(\mathcal{T} + H^2) + \frac{u^2}{4} \mathcal{T} \left(\frac{3}{2} \mathcal{T} + H^2 \right) - u E_r H \alpha, \quad (3.7a)$$

$$-2\alpha\gamma = 2\Lambda_{\text{eff}} - V(\mathcal{T} - E_\varphi^2) + \frac{u^2}{4} \mathcal{T} \left(\frac{3}{2} \mathcal{T} - E_\varphi^2 \right) - u E_r H (\gamma - \alpha), \quad (3.7b)$$

$$-2(\alpha' B + \alpha^2) = 2\Lambda_{\text{eff}} - V(\mathcal{T} - E_r^2) + \frac{u^2}{4} \mathcal{T} \left(\frac{3}{2} \mathcal{T} - E_r^2 \right) + u E_r H \gamma, \quad (3.7c)$$

where $\Lambda_{\text{eff}} := q - p^2/4$. These equations are invariant under the *duality mapping*

$$\begin{aligned}\alpha &\rightarrow \gamma, & \gamma &\rightarrow \alpha, \\ E_r &\rightarrow iH, & H &\rightarrow iE_r,\end{aligned}\tag{3.8}$$

which defines a useful correspondence between different solutions. The duality mapping has the same form as in GR _{Λ} [14].

In the case $u = 0$, one immediately concludes that

$$E_r E_\varphi = 0, \quad E_r H = 0, \quad E_\varphi H = 0.$$

This is a specific no-go theorem, which holds in GR _{Λ} [14]; it states that configurations with two nonvanishing components of the Maxwell field are dynamically not allowed.

Let us now return to the general case with $u \neq 0$. Analyzing the above gravitational field equations, one obtains the general no-go theorem (appendix B):

$$E_r E_\varphi H = 0.\tag{3.9}$$

- In any static and spherically symmetric configuration, it is dynamically impossible to have three nonvanishing components of the Maxwell field.

The theorem implies that at least one component of the Maxwell field has to vanish. Motivated by this result, we now turn our attention to exploring static and spherically symmetric solutions associated with an *electric* Maxwell field, specified by $H = 0$.

4 Dynamics in the electric sector

The electric sector of the Maxwell field is defined by

$$F = E_r b^0 b^1 + E_\varphi b^2 b^0.$$

The field equations are significantly simplified. The nondiagonal equations take the form

$$0 = -u \left(-\frac{1}{2} \gamma E_r^2 + \frac{1}{2} E_\varphi E'_\varphi B + \alpha E_\varphi^2 \right),\tag{4.1a}$$

$$0 = u E_r E_\varphi \alpha,\tag{4.1b}$$

$$V E_r E_\varphi - \frac{u^2}{4} E_r E_\varphi \mathcal{T} = 0,\tag{4.1c}$$

$$0 = -u E_r (E'_\varphi B + \gamma E_\varphi),\tag{4.1d}$$

$$0 = u \left(\frac{1}{2} \gamma E_r^2 + \frac{1}{2} E_\varphi E'_\varphi B + \gamma E_\varphi^2 \right),\tag{4.1e}$$

while the diagonal ones are:

$$-2(\gamma' B + \gamma^2) = 2\Lambda_{\text{eff}} - V\mathcal{T} + \frac{3u^2}{8} \mathcal{T}^2,\tag{4.2a}$$

$$-2\alpha\gamma = 2\Lambda_{\text{eff}} - V(\mathcal{T} - E_\varphi^2) + \frac{u^2}{4} \mathcal{T} \left(\frac{3}{2} \mathcal{T} - E_\varphi^2 \right),\tag{4.2b}$$

$$-2(\alpha' B + \alpha^2) = 2\Lambda_{\text{eff}} - V(\mathcal{T} - E_r^2) + \frac{u^2}{4} \mathcal{T} \left(\frac{3}{2} \mathcal{T} - E_r^2 \right).\tag{4.2c}$$

The analysis of these equations leads to the conclusion that the only interesting configuration is the one defined by the azimuthal electric field E_φ .

$E_r E_\varphi \neq 0$

In this case, equations (4.1b) and (4.1c) imply

$$\alpha = 0, \quad V - \frac{u^2}{4}\mathcal{T} = 0,$$

while (4.2c) yields

$$\Lambda_{\text{eff}} + \left(\frac{V}{u}\right)^2 = 0.$$

Using $V = v + up/2$ and the identity $ap + \alpha_3 q + \alpha_4 = 0$, the last relation leads to $\alpha_3 \alpha_4 - a^2 = 0$, which is in contradiction with the property $\Delta \neq 0$, adopted in section 2. Hence, our assumption $E_r E_\varphi \neq 0$ cannot be true, i.e. at least one of the components E_r, E_φ must vanish.

$E_r \neq 0, E_\varphi = 0$

In this case, there exists only the radial electric field. Equation (4.1a) yields $\gamma = 0$, which implies

$$K = \text{const.} \quad \Rightarrow \quad E_r = \text{const.}$$

Equation (4.2a) determines the value of $\mathcal{T} = E_r^2/2$. Combining (4.2a) and (4.2c), we obtain

$$\alpha' + \alpha^2 = -2\Lambda_{\text{eff}} - \frac{u^2}{8}\mathcal{T}^2 =: \kappa^2,$$

where the radial coordinate is chosen so that $B = 1$, for simplicity. The solution of this equation depends on the sign of κ^2 .

- a) For $\kappa^2 > 0$, we have $N = C \cosh \kappa(r + r_0)$, where r_0 and C are integration constants.
- b) For $\kappa^2 = 0$, we find $N = C(r + r_0)$.
- c) For $\kappa^2 < 0$ we find $N = C \sin |\kappa|(r + r_0)$.

All of these solutions are characterized by a constant electric field, hence, they are of no physical interest. As we shall see in the next section, the last case, defined by a sole azimuthal electric field, leads to a nontrivial dynamical situation.

5 Solution with azimuthal electric field

Since the Maxwell field equations do not impose any restriction on the azimuthal electric field, it is completely determined by the gravitational field equations.

The non-diagonal gravitational field equations are very simple:

$$E'_\varphi B + 2\alpha E_\varphi = 0, \quad E'_\varphi B + 2\gamma E_\varphi = 0. \quad (5.1a)$$

They imply

$$\begin{aligned}\alpha = \gamma &\Rightarrow N = C_1 K, \\ E_\varphi K^2 &= Q_2,\end{aligned}\tag{5.1b}$$

where C_1 and Q_2 are the integration constants.

In the set of the diagonal field equations (4.2), the first two take the form

$$-2(\gamma' B + \gamma^2) = 2\Lambda_{\text{eff}} - V\mathcal{T} + \frac{3u^2}{8}\mathcal{T}^2,\tag{5.2a}$$

$$-2\alpha\gamma = 2\Lambda_{\text{eff}} + V\mathcal{T} - \frac{u^2}{8}\mathcal{T}^2,\tag{5.2b}$$

while the third one is equivalent to (5.2a), since $\alpha = \gamma$. It is convenient to fix the radial coordinate by choosing

$$K = r \Rightarrow E_\varphi = \frac{Q_2}{r^2},\tag{5.3}$$

whereupon the field equations take the form

$$B^2 = -\Lambda_{\text{eff}} r^2 - V \frac{Q_2^2}{4r^2} + \frac{u^2}{64} \frac{Q_2^4}{r^6},\tag{5.4a}$$

$$BB' = -r\Lambda_{\text{eff}} + V \frac{Q_2^2}{4r^3} - \frac{3u^2}{64} \frac{Q_2^4}{r^7}.\tag{5.4b}$$

These two equations are consistent with each other, and they determine B^2 .

The above expressions for N, B, K and E_φ represent a complete solution describing the azimuthal electric field in 3D gravity with torsion, for any value of Λ_{eff} . In what follows, we shall restrict our considerations to the case of negative Λ_{eff} ,

$$\Lambda_{\text{eff}} = -\frac{1}{\ell^2},$$

which corresponds asymptotically to an AdS configuration, suitable for the calculation of the conserved charges. Using $C_1 = 1/\ell$, the solution can be written in the form

$$\begin{aligned}b^0 &= \frac{r}{\ell} dt, & b^1 &= B^{-1} dr, & b^2 &= r d\varphi, \\ \tilde{\omega}^0 &= -B d\varphi, & \tilde{\omega}^1 &= 0, & \tilde{\omega}^2 &= -\frac{B}{\ell} dt, \\ t^0 &= -\frac{Q_2^2}{4\ell r^3} dt, & t^1 &= \frac{3Q_2^2 B}{4r^4} dr, & t^2 &= -\frac{Q_2^2}{4r^3} d\varphi, \\ \omega^i &= \tilde{\omega}^i + \frac{1}{2} (pb^i + ut^i),\end{aligned}\tag{5.5a}$$

and

$$F = \frac{Q_2}{\ell} d\varphi dt.\tag{5.5b}$$

For $Q_2 = 0$, the solution reduces to the black hole vacuum.

Conserved charges

A deeper insight into the nature of the exact solution (5.5) is achieved by calculating the values of its conserved charges.

We start by choosing the *asymptotic conditions* at spatial infinity so that (i) the fields b^i , ω^i and F belong to the family (5.5), parametrized by Q_2 , and (ii) the corresponding asymptotic symmetries have well-defined canonical generators.

According to (i), the asymptotic form of the fields reads:

$$\begin{aligned} b^i{}_\mu &\sim \begin{pmatrix} \frac{r}{\ell} & 0 & 0 \\ 0 & \frac{\ell}{r} + V \frac{\ell^3 Q_2^2}{8r^5} & 0 \\ 0 & 0 & r \end{pmatrix}, \\ \omega^i{}_\mu &\sim \begin{pmatrix} \frac{pr}{2\ell} - \frac{uQ_2^2}{8\ell r^3} & 0 & -\frac{r}{\ell} + V \frac{\ell Q_2^2}{8r^3} \\ 0 & \frac{p\ell}{2r} + \frac{3uQ_2^2}{8\ell r^3} & 0 \\ -\frac{r}{\ell^2} + V \frac{Q_2^2}{8r^3} & 0 & \frac{pr}{2} - \frac{uQ_2^2}{8\ell r^3} \end{pmatrix}, \\ F &\sim \frac{Q_2}{\ell} d\varphi dt. \end{aligned} \tag{5.6}$$

Gauge symmetries of the theory are local translations, local Lorentz rotations and local $U(1)$ transformations, parametrized by ξ^μ , θ^i and λ , respectively. The subset of gauge transformations that respects the adopted asymptotic conditions is defined by the *asymptotic parameters*, which have the following form:

$$\xi^\mu = (\ell T_0, 0, S_0), \quad \theta^i = (0, 0, 0), \quad \lambda = \lambda(t, r, \varphi). \tag{5.7}$$

Here T_0 and S_0 are constant parameters associated with the rigid time translations and axial rotations, respectively, while the $U(1)$ parameter λ remains local, since the asymptotic conditions (5.6) do not restrict its form.

We are now ready to calculate the asymptotic charges by using the standard canonical approach. Technical details leading to the construction of the canonical generator G are summarized in appendix C. Since G acts on basic dynamical variables via the Poisson brackets, it must be differentiable. If this is not the case, the form of G can be improved by adding a suitable surface term. However, it turns out that under the adopted asymptotic conditions (5.6), G is differentiable automatically, without adding any surface term. Consequently, the canonical charges corresponding to the asymptotic parameters (5.7), the energy E , the angular momentum M and the electric charge Q of the solution (5.5), vanish:

$$E = 0, \quad M = 0, \quad Q = 0. \tag{5.8}$$

The vanishing of the electric charge is an expected result, since the azimuthal electric does not produce the radial flux. The vanishing of E and M can be understood by noting that the subleading terms in the asymptotic formulas (5.6) are "too small" to produce any nontrivial contribution to the corresponding surface integrals (a nontrivial contribution would be produced by the subleading terms of order r^{-1}).

Geometric structure

The metric of the solution (5.5) reads:

$$ds^2 = \frac{r^2}{\ell^2} dt^2 - \frac{dr^2}{\frac{r^2}{\ell^2} - V \frac{Q_2^2}{4r^2} + \frac{u^2}{64} \frac{Q_2^4}{r^6}} - r^2 d\varphi^2.$$

In the limit $u \rightarrow 0$, the metric reduces to the form found by Cataldo in GR_Λ [15], while for $Q_2 = 0$, it coincides with the black hole vacuum.

The scalar Cartan curvature is singular at $r = 0$:

$$R = -\varepsilon^{imn} R_{imn} = -6q + \frac{Q_2^2}{2r^4}.$$

Let us now show that the above solution is regular in the region $r > 0$. We begin by writing $B^2(r) = f(r^4)/r^6$, where

$$f(x) := \frac{x^2}{\ell^2} - V \frac{Q_2^2}{4} x + \frac{u^2 Q_2^4}{64}.$$

For large r , the function B^2 is positive. The coordinate singularities are related to the existence of zeros of $f(x)$ for some real and positive x . Equation $f(x) = 0$ has real and positive roots if the following two conditions are fulfilled:

$$V^2 - \frac{u^2}{\ell^2} \geq 0, \quad V > 0.$$

Using the relations

$$V^2 - \frac{u^2}{\ell^2} = -\frac{u^2}{\alpha_3^2} \Delta, \quad V \mp \frac{u}{\ell} = \frac{1}{24\pi\ell\Delta} c^\mp,$$

where c^\mp are classical central charges [7], and recalling that $\Delta \neq 0$, the above conditions imply

$$\begin{aligned} \Delta < 0 &\Leftrightarrow c^- c^+ > 0, \\ \frac{1}{\Delta}(c^- + c^+) > 0 &\Rightarrow c^- + c^+ < 0. \end{aligned}$$

By combining these relations, one concludes that both central charges have to be negative. However, the positivity of the BTZ black hole energy, obtained in the supersymmetric 3D gravity with torsion [6], implies that both c^- and c^+ are positive. Thus, $f(x)$ has no real and positive zeros, and consequently, $B^2(r) > 0$ for all $r > 0$.

- For $Q_2 \neq 0$, the solution (5.5) is regular in the region $r > 0$ and does not have the black hole structure.

Although the central charges c^\mp are originally defined by the asymptotic structure of the AdS sector of spacetime, here, in the presence of the azimuthal electric field, they have a new dynamical role, embedded in the geometric properties of the solution (5.5). The same phenomenon is observed also in the solution with self-dual Maxwell field [16].

6 Concluding remarks

In this paper, we studied exact solutions of the system of electric field coupled to 3D gravity with torsion.

(1) First, we proved the following theorem: in static and spherically symmetric configurations, the Maxwell field cannot have all three components different from zero.

(2) Motivated by this result, we restricted our consideration to the electric sector of the theory and found an exact solution with azimuthal electric field. This is the only nontrivial solution in this sector, and it represents a generalization of the Cataldo solution, found earlier in Riemannian GR_Λ . The values of its energy, angular momentum and electric charge all vanish, and its geometric structure is strongly influenced by the classical central charges, associated with the asymptotic AdS structure of spacetime.

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A The field equations

In this appendix, we present some technical details regarding the structure of the field equations (2.7b).

In Schwarzschild-like coordinate (t, r, φ) , the form of the static and spherically symmetric triad field is defined in (3.1). After calculating the Levi-Civita connection,

$$\tilde{\omega}^0 = -\gamma b^2, \quad \tilde{\omega}^1 = 0, \quad \tilde{\omega}^2 = -\alpha b^0,$$

where $\alpha = BN'/N$, $\gamma = BK'/K$, we find the Riemannian curvature:

$$\begin{aligned} \tilde{R}_0 &= -(\gamma' B + \gamma^2) b^1 b^2, & \tilde{R}_1 &= -\alpha \gamma b^2 b^0, \\ \tilde{R}_2 &= -(\alpha' B + \alpha^2) b^0 b^1. \end{aligned}$$

In the next step, we calculate the expressions $B_i = \varepsilon_{imn} t^m b^n$, $C_i = \varepsilon_{imn} t^m t^n$ and $\tilde{\nabla} t_i$:

$$\begin{aligned} B_0 &= -E_r H b^0 b^1 + (\mathcal{T} + H^2) b^1 b^2 - E_\varphi H b^2 b^0, \\ B_1 &= -E_r E_\varphi b^0 b^1 + E_\varphi H b^1 b^2 + (\mathcal{T} - E_\varphi^2) b^2 b^0, \\ B_2 &= (\mathcal{T} - E_r^2) b^0 b^1 + E_r H b^1 b^2 - E_r E_\varphi b^2 b^0, \end{aligned}$$

$$\begin{aligned} C_0 &= E_r H \mathcal{T} b^0 b^1 - \mathcal{T} \left(\frac{3}{2} \mathcal{T} + H^2 \right) b^1 b^2 + E_\varphi H \mathcal{T} b^2 b^0, \\ C_1 &= E_r E_\varphi \mathcal{T} b^0 b^1 - E_\varphi H \mathcal{T} b^1 b^2 - \mathcal{T} \left(\frac{3}{2} \mathcal{T} - E_\varphi^2 \right) b^2 b^0, \\ C_2 &= -\mathcal{T} \left(\frac{3}{2} \mathcal{T} - E_r^2 \right) b^0 b^1 - E_r H \mathcal{T} b^1 b^2 + E_r E_\varphi \mathcal{T} b^2 b^0, \end{aligned}$$

$$\begin{aligned}
\tilde{\nabla}t_0 &= \left(-\frac{1}{2}\gamma E_r^2 + \frac{1}{2}E_\varphi E'_\varphi B + \alpha E_\varphi^2 - \frac{1}{2}\alpha H^2 \right) b^0 b^1 + E_r H \alpha b^1 b^2 - E_r E_\varphi \alpha b^2 b^0, \\
\tilde{\nabla}t_1 &= H(E'_\varphi B + \alpha E_\varphi) b^0 b^1 - E_r(E'_\varphi B + \gamma E_\varphi) b^1 b^2 + E_r H(\gamma - \alpha) b^2 b^0, \\
\tilde{\nabla}t_2 &= -E_r H \gamma b^0 b^1 + \left(\frac{1}{2}\gamma E_r^2 + \frac{1}{2}E_\varphi E'_\varphi B + \gamma E_\varphi^2 + \frac{1}{2}\alpha H^2 \right) b^1 b^2 - E_\varphi H \gamma b^2 b^0.
\end{aligned}$$

The above results completely determine the Cartan curvature (2.7c), and lead to the second gravitational field equation in the form (3.6) and (3.7).

B The proof of $E_r E_\varphi H = 0$

In this appendix, we prove the general no-go theorem formulated at the end of section 3.

We begin by assuming $E_r E_\varphi H \neq 0$. Then, the consistency of the first three and the last three nondiagonal equations in (3.6) is ensured by a single relation:

$$E'_\varphi B + E_\varphi(\alpha + \gamma) = 0 \quad \Rightarrow \quad E_\varphi N K = C_1, \quad (\text{B.1})$$

where C_1 is a constant. With (B.1), the set of equations (3.6) reduces to:

$$\begin{aligned}
V E_r H - \frac{u^2}{4} E_r H \mathcal{T} &= u \left(-\frac{1}{2}\gamma E_r^2 + \frac{1}{2}\alpha E_\varphi^2 - \frac{1}{2}\gamma E_\varphi^2 - \frac{1}{2}\alpha H^2 \right), \\
V E_\varphi H - \frac{u^2}{4} E_\varphi H \mathcal{T} &= -u E_r E_\varphi \alpha, \\
V E_r E_\varphi - \frac{u^2}{4} E_r E_\varphi \mathcal{T} &= -u E_\varphi H \gamma,
\end{aligned} \quad (\text{B.2})$$

The consistency of these equations implies:

$$\alpha E_r^2 = \gamma H^2, \quad (\gamma - \alpha) \mathcal{T} = 0, \quad (\text{B.3})$$

which yields either $\alpha = \gamma$ or $\mathcal{T} = 0$. Now, we use this result to explore the diagonal equations (3.7).

(a) If $\alpha = \gamma$, then we have $E_r = \epsilon H$, $\epsilon = \pm 1$, equations (B.2) and (3.7b) yield

$$V - \frac{u^2}{4} \mathcal{T} = -\epsilon u \alpha, \quad \Lambda_{\text{eff}} + \left(\epsilon \alpha - \frac{u \mathcal{T}}{4} \right)^2 = 0,$$

and consequently,

$$\Lambda_{\text{eff}} + \left(\frac{V}{u} \right)^2 = 0. \quad (\text{B.4})$$

This relation leads to $\alpha_3 \alpha_4 - a^2 = 0$, in contradiction with the assumption $\Delta \neq 0$, adopted in section 2. Thus, $\alpha = \gamma$ is not allowed.

(b) If $\mathcal{T} = 0$, equations (B.2) imply

$$V H = -u E_r \alpha, \quad V E_r = -u H \gamma,$$

leading to $V^2 = u^2 \alpha \gamma$. Then, (3.7b) reduces to $\alpha \gamma = -\Lambda_{\text{eff}}$, and we obtain again (B.4), so that $\mathcal{T} = 0$ is also not allowed.

This completes the proof that the configuration $E_r E_\varphi H \neq 0$ can not be realized dynamically, which is exactly the content of the no-go theorem.

C Canonical generator

In this appendix, we construct the canonical generator of gauge transformations.

Starting with the Lagrangian variables $(b_\mu^i, \omega_\mu^i, A_\mu)$ and the related canonical momenta $(\pi_\mu^i, \Pi_\mu^i, \pi^\mu)$, we find the following primary constraints:

$$\begin{aligned}\phi_i^0 &:= \pi_i^0 \approx 0, & \phi_i^\alpha &:= \pi_i^\alpha - \alpha_4 \varepsilon^{0\alpha\beta} b_{i\beta} \approx 0, \\ \Phi_i^0 &:= \Pi_i^0 \approx 0, & \Phi_i^\alpha &:= \Pi_i^\alpha - \varepsilon^{0\alpha\beta} (2ab_{i\beta} + \alpha_3 \omega_{i\beta}) \approx 0, \\ \phi &:= \pi^0 \approx 0,\end{aligned}$$

The canonical Hamiltonian is linear in unphysical variables, as expected:

$$\begin{aligned}\mathcal{H}_c &= b^i_0 \mathcal{H}_i + \omega^i_0 \mathcal{K}_i - A_0 \partial_\alpha \pi^\alpha + \partial_\alpha D^\alpha, \\ \mathcal{H}_i &= -\varepsilon^{0\alpha\beta} \left(a R_{i\alpha\beta} + \alpha_4 T_{i\alpha\beta} - \Lambda \varepsilon_{ijk} b^j_\alpha b^k_\beta - \frac{1}{2} \Theta_{i\alpha\beta} \right), \\ \mathcal{K}_i &= -\varepsilon^{0\alpha\beta} \left(a T_{i\alpha\beta} + \alpha_3 R_{i\alpha\beta} + \alpha_4 \varepsilon_{ijk} b^j_\alpha b^k_\beta \right), \\ D^\alpha &= \varepsilon^{0\alpha\beta} \left[\omega^i_0 (2ab_{i\beta} + \alpha_3 \omega_{i\beta}) + \alpha_4 b^i_0 b_{i\beta} \right] + A_0 \pi^\alpha.\end{aligned}$$

Going over to the total Hamiltonian,

$$\mathcal{H}_T = \mathcal{H}_c + u^i_\mu \phi_i^\mu + v^i_\mu \Phi_i^\mu + w \pi^0,$$

we find that the consistency conditions of the primary constraints π_i^0 , Π_i^0 and π^0 yield the secondary constraints:

$$\mathcal{H}_i \approx 0, \quad \mathcal{K}_i \approx 0, \quad \partial_\alpha \pi^\alpha \approx 0.$$

The consistency of the additional primary constraints ϕ_i^α and Φ_i^α , leads to the determination of the multipliers u^i_α and v^i_α , resulting in the final form of the total Hamiltonian:

$$\begin{aligned}\mathcal{H}_T &= \hat{\mathcal{H}}_T + \partial_\alpha \hat{D}^\alpha, \\ \hat{\mathcal{H}}_T &= b^i_0 \hat{\mathcal{H}}_i + \omega^i_0 \hat{\mathcal{K}}_i - A_0 \partial_\alpha \pi^\alpha + u^i_0 \pi_i^0 + v^i_0 \Pi_i^0 + w \pi^0,\end{aligned}$$

where

$$\begin{aligned}\hat{\mathcal{H}}_i &= \mathcal{H}_i - \nabla_\beta \phi_i^\beta + \varepsilon_{ijk} b^j_\beta (p \phi^{k\beta} + q \Phi^{k\beta}) + (u \phi^{j\beta} - v \Phi^{j\beta}) \frac{1}{2} \Theta_{ji\beta}, \\ \hat{\mathcal{K}}_i &= \mathcal{K}_i - \nabla_\beta \Phi_i^\beta - \varepsilon_{ijk} b^j_\beta \phi^{k\beta}, \\ \hat{D}^\alpha &= D^\alpha + b^i_0 \pi_n^\alpha + \omega^i_0 \Pi_n^\alpha.\end{aligned}$$

The consistency conditions of the secondary constraints are identically satisfied, and the Hamiltonian consistency procedure is thereby completed.

Regarding the classification of constraints, we note that $(\pi_i^0, \Pi_i^0, \pi^0)$ and $(\hat{\mathcal{H}}_i, \hat{\mathcal{K}}_i, \partial_\alpha \pi^\alpha)$ are first class, while $(\phi_i^\alpha, \Phi_i^\alpha)$ are second class.

Using the well-established Castellani procedure [15], we obtain the canonical generator:

$$\begin{aligned}G &= -G_1 - G_2 - G_3, \\ G_1 &:= \dot{\xi}^\rho (b^i_\rho \pi_i^0 + \omega^i_\rho \Pi_i^0 + \pi^0 A_\rho) \\ &\quad + \xi^\rho \left[b^i_\rho \hat{\mathcal{H}}_i + \omega^i_\rho \hat{\mathcal{K}}_i + (\partial_\rho b^i_0) \pi_i^0 + (\partial_\rho \omega^i_0) \Pi_i^0 + (\partial_\rho A_0) \pi^0 \right], \\ G_2 &:= \dot{\theta}^i \Pi_i^0 + \theta^i \left[\hat{\mathcal{K}}_i - \varepsilon_{ijk} (b^j_0 \pi^{k0} + \omega^j_0 \Pi^{k0}) \right], \\ G_3 &:= \dot{\lambda} \pi^0 - \lambda \partial_\alpha \pi^\alpha,\end{aligned}$$

where the integration symbol $\int d^2x$ is omitted for simplicity. The action of G on the fields is defined by the Poisson bracket operation, $\bar{\delta}\phi := \{\phi, G\}$. As one can verify, $\bar{\delta}\phi$ coincides with the combination of Poincaré plus $U(1)$ gauge transformations on shell.

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